Reciprocity laws and torsion classes

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Motivation

Perfect squares and quadratic residues

Question

What are the last digits of perfect squares?

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225...

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Harder question

Let $\ell \geq 5$ be a prime number. Can 3 be the last digit of a perfect square in base ℓ? When does the polynomial

$$x^2 - 3$$

split into two distinct linear factors modulo ℓ?

The law of quadratic reciprocity

Let p and ℓ be distinct odd primes. The law of quadratic reciprocity relates whether

- p is a quadratic residue modulo ℓ

to whether

- ℓ is a quadratic residue modulo p.

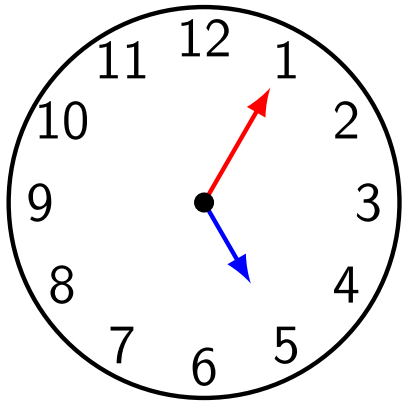
$$\left(\frac{p}{\ell}\right) \cdot \left(\frac{\ell}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{\ell-1}{2}}$$

It was conjectured by Euler and Legendre and first proved by Gauss in 1796.

The law of quadratic reciprocity

Consequence: whether 3 can be the last digit of a perfect square in base ℓ only depends on ℓ modulo $3 \cdot 4 = 12$.

- For ℓ equal to 13, 37, 61 and 1093, the answer is "Yes".
- For ℓ equal to 17, 29, 41 and 1637, the answer is "No".



Higher-dimensional reciprocity laws

The infinite product

$$f(q) := q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \dots$$

and the Diophantine equation

$$E : y^2 + y = x^3 - x^2$$

seem to know about each other in a mysterious way.

Higher-dimensional reciprocity laws

- The coefficient a_ℓ of q^ℓ in the expansion of $f(q)$ takes the values:

ℓ	2	3	5	7	13	17	19	23	29
a_ℓ	-2	-1	1	-2	4	-2	0	-1	0

- The number N_ℓ of solutions to $y^2 + y \equiv x^3 - x^2 \pmod{\ell}$ takes the values:

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- We always seem to have $a_\ell = \ell - N_\ell$.

Modular forms and elliptic curves

- The power series $f(q)$ is the Fourier expansion of a modular form.
- The Diophantine equation E represents an elliptic curve over $\mathbb{Q}$.

The reciprocity law

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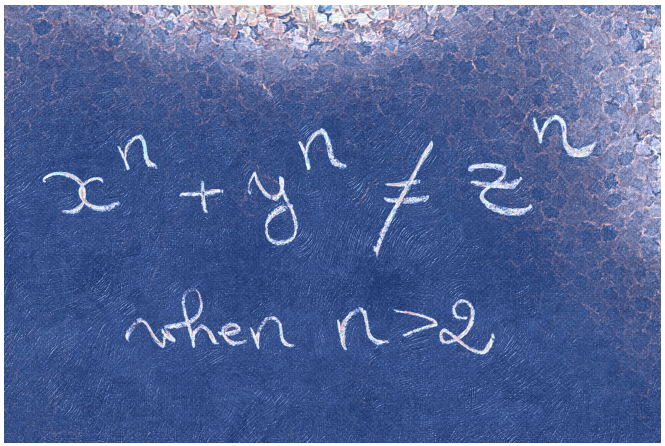

What is the Langlands correspondence?

The global Langlands correspondence matches:

$$\{\text{spectral data}\} - \text{seen on the automorphic side}$$

with

$$\{\text{arithmetic data}\} - \text{seen on the Galois side.}$$

To describe such a correspondence, we first need to choose:

- a global field F, such as $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{3})$ or $\mathbb{Q}(i)$;
- a connected reductive group G, such as GL_n, or SL_n, or Sp_{2n}.

The Galois side

Given a number field F, we are interested in its absolute Galois group

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For example, if $E/\mathbb{Q}$ is an elliptic curve, we have an action of $\Gamma_{\mathbb{Q}}$ on the torsion points

$$E[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^2.$$

As n varies, these give rise to a Galois representation

$$\rho_E : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p).$$

More generally, the étale cohomology of algebraic varieties defined over number fields is a source of Galois representations.

The automorphic side

A modular form is a holomorphic function on the upper-half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$$

that satisfies many symmetries and a growth condition.

There is a transitive action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$:

$$z \mapsto \frac{az+b}{cz+d} \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

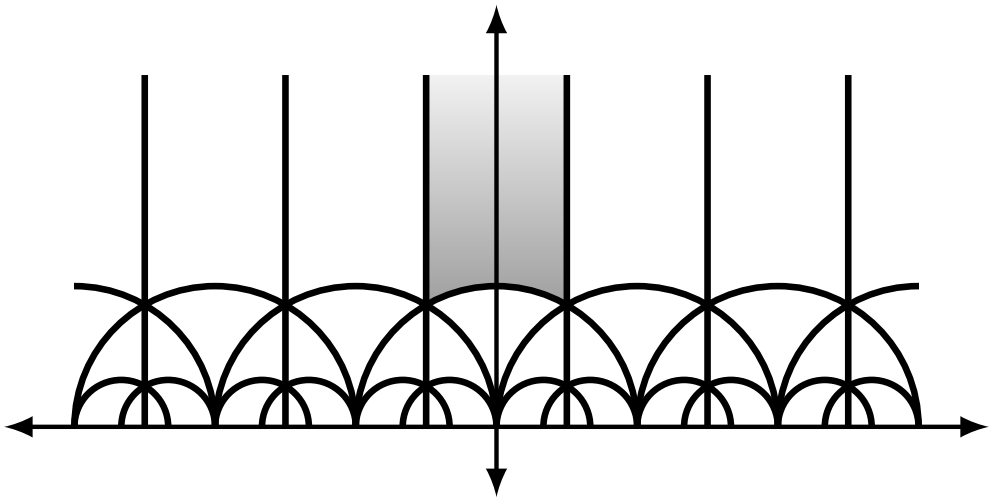
This gives the identification of $\mathbb{H}$ with the symmetric space

$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$$

for the group SL_2.

Modular curves

Modular curves are quotients $\Gamma\backslash\mathbb{H}^2$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup.



The quotients $\Gamma\backslash\mathbb{H}^2$ arise from algebraic curves defined over number fields. Their étale cohomology is a source of Galois representations

$$f \mapsto \rho_f : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p).$$

The correspondence

Recall the example

$$E : y^2 + y = x^3 - x^2, \text{ and}$$

$$f(z) := q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2, q = e^{2\pi i z}.$$

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When we say that E and f are matched under the global Langlands correspondence, we mean that there is an isomorphism

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This implies the explicit reciprocity law

$$a_\ell = \ell - N_\ell \text{ for all primes } \ell \neq 11,$$

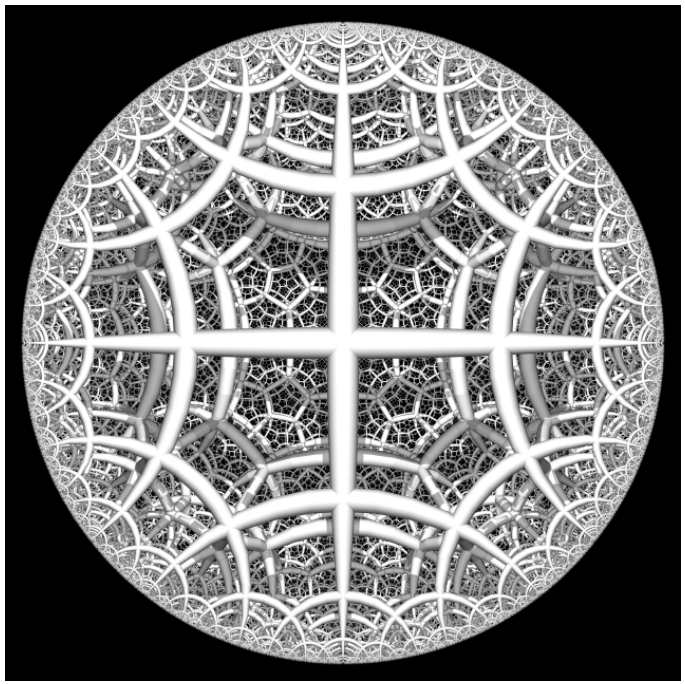
by taking traces at distinguished conjugacy classes $\{\mathrm{Frob}_\ell\} \in \Gamma_{\mathbb{Q}}$.

New frontiers

Arithmetic hyperbolic 3-manifolds

Let F be an imaginary quadratic field. The symmetric space for SL_2/F is hyperbolic 3-space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R}) \simeq \mathbb{H}^3$.

If $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$ is a sufficiently small congruence subrgroup, the $\Gamma\backslash\mathbb{H}^3$ are arithmetic hyperbolic 3-manifolds. They do not have the structure of algebraic varieties!



Torsion cohomology

- Let F be a CM field and X_Γ be a locally symmetric space for GL_n/F.

Theorem 1 (Scholze, 2013)

Any system of Hecke eigenvalues occurring in $H^*(X_\Gamma, \mathbb{F}_\ell)$ has an associated Galois representation $\rho : \Gamma_F \to \mathrm{GL}_n(\mathbb{F}_\ell)$.

- This strengthens previous work of Harris–Lan–Taylor–Thorne for $H^*(X_\Gamma, \mathbb{Q}_\ell)$ (automorphic forms).
- The need to consider torsion classes comes from the Calegari–Geraghty extension of the Taylor–Wiles method.

The Sato–Tate conjecture

Recall the elliptic curve

$$y^2 + y = x^3 - x^2.$$

By Hasse, the normalised error terms $\frac{\ell - N_\ell}{2\sqrt{\ell}}$ are in $[-1, 1]$. The Sato–Tate conjecture predicts they are equidistributed with respect to the semicircle probability measure $\frac{2}{\pi}\sqrt{1-x^2}dx$.

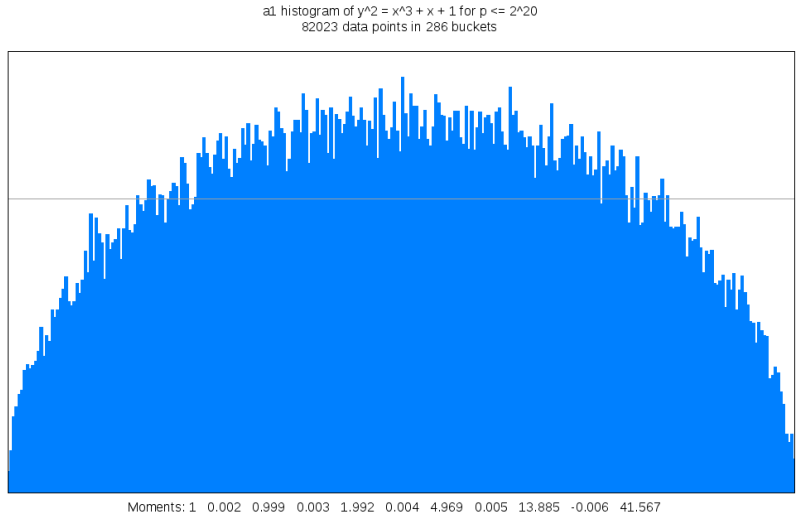


Image credit: A. Sutherland, "Sato–Tate Distributions"

The Sato–Tate conjecture

- For (most) elliptic curves over $\mathbb{Q}$, the Sato–Tate conjecture was proved by Clozel, Harris, Shepherd-Barron, and Taylor in 2008.

Theorem 2 (Allen, Calegari, C., Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne, 2018)

Let F be a CM field and let E/F be an elliptic curve without complex multiplication. Then E is potentially automorphic and satisfies the Sato–Tate conjecture.

The Ramanujan conjecture

- The Ramanujan conjecture bounds the absolute value of Fourier coefficients of modular eigenforms.
- Its generalisation predicts that the local components of cuspidal automorphic representations of GL_n are tempered.

Theorem 3 (Allen, Calegari, C., Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne, 2018)

Let F be a CM field and π be a cuspidal automorphic representation of GL_2/F of parallel weight 2. Then π satisfies the generalised Ramanujan–Petersson conjecture.

Shimura varieties

- Theorems 2 and 3 rely on the following result about Shimura varieties.

Theorem 4 (C., Scholze, 2015, 2019, Koshikawa 2021)

Let X_K be a unitary Shimura variety and $\mathfrak{m}$ be a system of Hecke eigenvalues occurring in $H^i_{(c)}(X_K, \mathbb{F}_p)$. Under mild technical assumptions on X_K, if $\mathfrak{m}$ is sufficiently generic, then

$$H^i_c(X_K, \mathbb{F}_p)_{\mathfrak{m}} = 0 \text{ for } i \geq \dim X_K.$$

- The proof relies on the geometry of the Hodge–Tate period morphism, which gives a new way to access the cohomology of Shimura varieties.

Local-global compatibility

Theorem 5 (C., Newton, in preparation)

Assume F is a CM field with $F^+ \neq \mathbb{Q}$. Under mild technical assumptions, the Galois representations constructed in Theorem 1 satisfy local-global compatibility at $\ell = p$ in the crystalline case.

- This goes significantly beyond the state of the art: p can be small, highly ramified in F, and the Hodge–Tate weights of the Galois representation can be arbitrarily high.

Thank you!